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Summary

Although orographic lee waves are probably common over the British Isles, their amplitude is so critically dependent on airstream characteristics and on the scale of local topography that they are likely to be insignificant unless a variety of conditions are satisfied. Precise criteria for large amplitude waves cannot be formulated for airstreams in general, but simplification of the general problem illustrates the pronounced effect of mountain size and shape, stability and wind speed on the lee wave amplitude.

1. Introduction

In theoretical studies of airflow over mountains a good deal of attention has been given to formulating and discussing conditions under which a system of atmospheric lee waves will form in an airstream downwind of hills or mountain ranges (see, for example, Scorer, 1949 and 1953). These conditions involve the vertical variation of a parameter usually denoted by l^2 and given by

$$l^2 = g\beta/u^2 - (1/u)(\partial^2 u/\partial z^2)$$

where g = acceleration due to gravity
β = stability factor defined as $(1/\theta)(d\theta/dz)$, θ being the potential temperature
u = component of wind across the mountain ridge
z = height.

Theoretical study has shown that an airstream can contain lee waves only if l^2 decreases sufficiently with height. In practice such a simple criterion is often difficult to exploit owing to the complex, irregular vertical distributions of l^2 usually found in real airstreams. It does appear, however, that lee wave conditions are frequently satisfied over the British Isles, especially during winter. But familiarity with aircraft reports has given us the impression that only a few of these theoretically possible waves have substantial effects on powered aircraft. This suggests that, although waves may be very common, their amplitude may often be so small that pilots scarcely notice their effects during flight. On some occasions, however, pilots have encountered waves of distinctly troublesome magnitudes.

Accordingly, a preliminary study of the factors controlling the lee wave amplitude has been made and this paper reveals the extreme sensitivity of such amplitude to the stability characteristics of the general airstream and to the local scale of mountain ranges in relation to the wavelength of the free oscillations of the airstream.

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2. Factors determining the lee wave amplitude

Scorer (1949) has derived expressions for the flow over an idealised two-dimensional ridge whose height, ζ , at a horizontal distance x from the crest is given by

$$\zeta = hb^2/(b^2 + x^2) \qquad (1)$$

where h is the height of the ridge and b is the "half-width" parameter. The flow pattern through lee waves induced by such a ridge is indicated by the term

$$\zeta_z = -2\pi h b e^{-kb} (u_1/u_z) \psi_{z,k} \left(\partial \psi_{1,k}/\partial k\right)^{-1} \sin kx \qquad (2)$$

where ζ_z = displacement of a streamline from its undisturbed level at height z
k = the lee wave number
u = horizontal wind speed
ψ = satisfies the equation

$$\partial^2\psi/\partial z^2 + (l^2 - k^2)\psi = 0 \qquad (3)$$

and the suffix 1 refers to ground level.

Thus the lee wave amplitude varies as

$$hbe^{-kb}, \quad u_1/u_z \quad \text{and} \quad \psi_{z,k}\left(\partial \psi_{1,k}/\partial k\right)^{-1}$$

It is convenient to consider these three factors separately.

3. The effect of mountain size and shape

So far as the mountain is concerned, the lee wave amplitude is proportional to hbe^{-kb}. A linear variation with mountain height is to be expected since the results are based on perturbation theory. The expression be^{-kb} has a maximum when $b = k^{-1}$; this means that if the mountain height remains constant the lee wave amplitude attains its greatest value when the mountain width is suitably adjusted to the natural wavelength of the airstream. The variation of lee wave amplitude with mountain width is illustrated by the dashed curve in Fig. 1. In this diagram k has been taken as 1 km^{-1}, i.e. the wavelength of the airstream is 2π km. The maximum at $b = k^{-1}$ is pronounced, and the amplitude falls off sharply for narrower or broader mountains. We may liken this effect to resonance; the natural wavelength is a function of the l^2 profile, i.e. of the airstream's stability and wind profiles, but the lee wave amplitude is likely to be small if the width of the mountain does not match the wavelength of the airstream.

We next enquire how the lee wave amplitude varies if we vary the mountain size while preserving its shape, i.e. the height and width are varied in the same proportion. On this basis the variation of lee wave amplitude has the same form as previously but the maximum amplitude now occurs when $b = 2k^{-1}$; the appropriate curve is drawn as a full line in Fig. 1. Thus we can regard the mountain for which $b = k^{-1}$ as having the optimum width for any specified height; if we then increase the overall mountain size the lee wave amplitude tends to increase with the height factor, h, and to decrease as the width departs from the optimum value. Keeping the ratio h:b constant allows the increasing height to dominate the amplitude tendency until $b = 2k^{-1}$; any further increase in overall mountain size produces a sharp decline in lee wave amplitude. Indeed we see, for example, that if a given airstream produces lee waves of a certain amplitude downstream of a mountain for which

/$b = 2k^{-1}$

$b = 2k^{-1}$ then the same airstream would only produce waves about 1/27 of that amplitude downstream of a mountain four times larger. Clearly this "resonance" effect is of considerable importance.

We infer that any large amplitude lee waves which happen to occur in an airstream will not necessarily be associated with the largest of the mountain ridges or hills in the underlying terrain. Thus an airstream which is favourable for significant waves over, say, the Cotswolds or the Chilterns will not necessarily produce even bigger waves over the Scottish Highlands or the Welsh Mountains. It is true that typical irregular terrain contains orographic features on all scales, but in any given region the dominant features tend to have a common scale.

In view of these conslusions it would be an advantage if the natural wavelength of an airstream could be calculated readily. But, although a system of equations determining the wavelength may formally be set up, the necessary computations are too laborious for routine work on real airstreams. We know, however, that the wavelength lies somewhere between the maximum and minimum values of $2\pi\ l^{-1}$ and we may say that in general if the winds are light at all levels the wavelength will be short, while strong winds will be associated with a long wavelength. Thus we deduce that the larger the mountains, the stronger are the wind speeds necessary for large amplitude lee waves, and such a deduction is supported by Förchtgott's observational studies made in Czechoslovakia (1949). The powerful "Bishop" wave of the Sierra Nevada in California develops only in strong winds approaching jet stream intensity; in this case the mountain ridge is a large feature requiring an airstream with a long natural wavelength for the maximum effect to be achieved.

4. Effect of the wind profile

The lee wave amplitude varies as u_1/u_z where u_1 is the surface wind speed and u_z the wind at height z. Consideration of this factor alone may give the impression that a decrease of wind with height would favour large amplitude lee waves. But we know that for waves to be possible at all l^2 must decrease with height. Such a decrease can be achieved by an increase of wind speed or by a decrease of stability with height, or by some combination of wind and stability distributions. We conclude that if we have two airstreams which, although having different wind and stability profiles, nevertheless have the same l^2 profiles, then the airstream in which u_1/u_z is larger will have the greater amplitude lee waves. Another way of looking at this is to note that, although an airstream with very light winds near the surface and much stronger winds higher up can contain lee waves, the amplitude of the waves would probably be small; other things being equal, larger amplitude waves would occur if the airstream satisfies the conditions for waves partly or mainly by stability variations, so that moderate or strong surface winds can be present without the necessary variation of l^2 with height being destroyed.

This viewpoint is consistent with the observational evidence from aircraft reports collected at Northolt Airport and discussed by Pilsbury (1955). In these reported waves marked stability was always present at middle levels while the low level wind speeds were usually more than 20 kt. and never less than 15 kt.

5. Effect of the l^2 profile

The lee wave amplitude also varies as $\psi_{z,k}\left(\partial\psi_{1,k}/\partial k\right)^{-1}$, which is a function of height and the parameter l^2. At one or more levels this amplitude factor attains some maximum value and we shall focus our attention on this maximum as a basis for studying the general effect of the l^2 profile. Variations in the l^2 profile will of course modify the lee wave number k and also, therefore, the factor be^{-kb}. However, in the present section we confine ourselves to examining the factor $\psi_{z,k}\left(\partial\psi_{1,k}/\partial k\right)^{-1}$ alone. For a given mountain the simultaneous variations of be^{-kb} would intensify the characteristics which emerge.

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For ease of mathematical manipulation we follow Scorer by considering the atmosphere to comprise:-

(a) an adiabatic layer ($l = 0$) from the ground to a height h_a.

(b) an intermediate layer in which $l = l_i$ between the levels h_a and $(h_a + h_i)$.

(c) an upper layer in which $l = l_s$ ($< l_i$) extending upwards from the level $(h_a + h_i)$.

Although this three layer structure is a compromise between reality and simplicity it is broadly in accordance with the type of airstream structure which obtains on occasions when aircraft report marked waves. In the first part of this discussion we carry the simplification further by putting $h_a = 0$. Then, taking the top of the intermediate layer as the zero of the height scale (i.e. the ground at $z = -h_i$) and assuming that waves must die out at high levels, the appropriate solutions of eq. (3) are

$$\left.\begin{aligned} \psi_s &= A \exp(-\mu_s z) \\ \psi_i &= A\{\cos \mu_i z - (\mu_s/\mu_i) \sin \mu_i z\} \end{aligned}\right\} \qquad (4)$$

where $\mu = \{|l^2 - k^2|\}^{\frac{1}{2}}$, the suffixes i and s refer to the lower and upper layers respectively, and A is a constant.

If we apply the condition that ψ_i must vanish at the ground downwind of an obstacle we obtain an equation determining the lee wave number, k. The equation may have a number of solutions, but, as Scorer (1949) has pointed out, the smallest root (corresponding to the longest wavelength) will dominate the flow. If the wave equation has n solutions we find that, at the n heights above the ground given by $h_n = \pi/2\mu_i, 3\pi/2\mu_i, \ldots\ldots (2n-1)\pi/2\mu_i$, the amplitude factor $\psi_{z,k}(\partial\psi_{i,k}/\partial k)^{-1}$ assumes a maximum value, a_n, given by

$$a_n = (\mu_i/k)(h_i + 1/\mu_s)^{-1} \qquad (5)$$

Mere inspection of this equation does not easily reveal how a_n varies with the airstream parameters l_i, l_s and h_i; this is because it is not possible to express k explicitly in terms of those parameters. It is simpler to study graphical representations of the equation and for this purpose the introduction of an additional dependent variable is helpful. Thus as k must lie between l_i and l_s we may write

$$k^2 = l_s^2 + L^2 \cos^2\theta \qquad (6)$$

$$L^2 = l_i^2 - l_s^2 \qquad (7)$$

where $0 \leq \theta \leq \pi/2$

We shall then have

$$\left.\begin{aligned} \mu_i &= L \sin\theta \\ \mu_s &= L \cos\theta \end{aligned}\right\} \qquad (8)$$

and the equation for the lee wavelength reduces to

$$(n\pi - \theta)/\sin\theta = Lh_i \qquad (9)$$

With this notation the lee wave amplitude factor at the levels where it is a maximum becomes

$$a_n = L^2(l_s^2 + L^2\cos^2\theta)^{-\frac{1}{2}}(n\pi - \theta + \tan\theta)^{-1}\sin^2\theta \qquad (10)$$

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This equation may be used to calculate the maximum amplitude for various values of l_s, L and Θ, the corresponding values of h_i being determined by eq.(9). Fig. 2 shows an isometric sketch in which the family of curves delineates a surface illustrating the way in which the maximum lee wave amplitude varies with l_s and h_i for a constant value of L ($=5$ km^{-1} in this figure). For values of h_i below the plane ABCD the condition for lee waves is not satisfied. A striking feature of the diagram is that as h_i is increased beyond this critical value we pass very rapidly from a condition in which there can be no lee waves to one in which they will have large amplitudes. It can also be demonstrated that if h_i were kept constant as L varied, the lee wave amplitude would behave similarly as a limiting value of L were crossed. It has been stated that the greater the decrease of l^2 with height, the more likely are lee waves because one can then be more confident that the condition for them is satisfied. Although this statement is correct it is liable to imply that the greater the decrease in l^2, the larger will be the amplitude of lee waves; casual interpretation of the perturbation restrictions may also suggest that deep layers having high values of l favour large amplitude waves. However, it appears that the optimum conditions, when the airstream is most sensitive, are very close to the critical conditions for waves, and that if conditions are made "more favourable" for waves to occur the amplitude falls off sharply. We also note that as l_s increases this sensitivity decreases somewhat (but remains important) and the amplitude falls off.

The variations illustrated in Fig. 2 are for a constant value of L of 5 km^{-1}. We next examine the effect of varying L, which is a measure of the decrease of l^2 with height. Fig. 3 shows curves for the lee wave amplitude (once again for the level at which it is a maximum) for L = 2,5 and 10 km^{-1}. For these curves the other variable is h_i; l_s is kept constant at 0.2 km^{-1} throughout. The corresponding wavelengths are depicted in the left hand section of the diagram. Full curves refer to n = 1, and dashed curves to n = 2; curves for higher values of n are not shown, but the tendency for the wave amplitudes to decrease as k acquires more possible values is maintained, and unless l_s is extremely small this decrease is a rapid one.

We see again the great sensitivity of the airstream near the critical values of the parameters for waves to occur; it is also apparent that the maximum amplitude becomes potentially greater as L is made larger. Indeed, it appears that the more the lower layer tends towards a sharp inversion, the greater does the maximum possible amplitude become. However, since the boundary conditions used at the interface between the two layers assume continuity of density we cannot extend our conclusions to an ideally sharp inversion without modifying these boundary conditions. Nevertheless the conclusion remains valid that greater amplitudes are possible if there is considerable static stability through a shallow layer than with smaller stability through a deep layer.

The variation of amplitude with L illustrated in Fig. 3 enables us to visualise a set of surfaces, one of which is given in Fig. 2, for a single value of L. We may then summarise the principal conclusions of Figs. 2 and 3 by saying that the largest amplitude lee waves will be possible when

(a) conditions are slightly more favourable for waves than the limiting conditions for them to occur at all. By "slightly more favourable" we mean that the stability or depth of the layer having the large l^2 should be a little greater than the limiting combination for waves.

(b) L should be large.

(c) l_s should be small.

We next consider briefly the maximum vertical velocity in these lee waves; its variation with the airstream parameters will not follow exactly that of the

amplitude because of the wavelength changes. Well away from the mountain the displacement of the streamline at the level where $\psi_{2,K}(\partial\psi_{1,K}/\partial K)^{-1}$ has a maximum will be given by putting $z = h_n$ in the equation

$$\zeta_z = -2\pi h_1 b e^{-Kb}(u_1/u_2)a_n \sin kx \qquad (11)$$

so that if w is the magnitude of the maximum vertical velocity,

$$w = \left[u_2(\partial\zeta_z/\partial x)\right]_{max} = 2\pi h_1 b e^{-Kb} u_1 a_n k \qquad (12)$$

Thus
$$w \propto \mu_i\left(h_i + 1/\mu_s\right)^{-1}$$

$$\propto L^2(n\pi - \theta + \tan\theta)^{-1}\sin^2\theta \qquad (13)$$

We note that eq. (13) is independent of l_s so that a family of curves is sufficient to illustrate the variation of vertical velocity with the three parameters defining the airstream. Such curves are drawn in Fig. 4 for $L = 1, 2, 3, 4, 5, 7$ and $10\ km^{-1}$ and for $n = 1$. These curves exhibit features which are similar to the amplitude curves, the main differences being that sensitivity is somewhat less critical near the optimum combination of h_i and L.

It is impracticable to extend the graphical representation of the maximum amplitude factor to deal with a three layer atmosphere fully, but we can study such a model to a limited extent. The addition of an adiabatically mixed layer, that is, a layer with neutral static stability, (denoted by suffix a) at the bottom requires an additional solution of eq. (3) having the form

$$\psi_a = B\cosh kz + C\sinh kz \qquad (14)$$

where B and C are constants. Further application of the boundary conditions leads to the following expression for the lee wave amplitude factor at the level where it is a maximum:

$$a_n = \left(\mu_i^2 - l_i^2\sin^2\nu\right)^{\frac{1}{2}}\left[k(h_i + 1/\mu_s) + \tfrac{1}{2}(l_i^2/Lk)\sin 2\nu\ \operatorname{cosec}\theta - (\mu_i^2 - l_i^2\sin^2\nu)h_a/k\right]^{-1} \qquad (15)$$

where

$$\tan\nu = (\mu_i/k)\tanh(kh_a) \qquad (16)$$

and
$$\nu = n\pi - \theta - Lh_i\sin\theta \qquad (17)$$

The condition for these equations to yield n real solutions for k can be derived as

$$Lh_i > \tfrac{1}{2}(2n-1)\pi - \alpha \qquad (18)$$

where

$$\alpha = \tan^{-1}\left[(L/l_s)\tanh(l_s h_a)\right] \qquad (19)$$

and
$$0 \geqslant \alpha \geqslant \pi/2$$

Eq. (18) may be regarded as the condition for lee waves to be possible in this three layer model and it corresponds to $Lh_i > \frac{1}{2}(2n-1)\pi$ for the two layer model; indeed as $h_a \to 0$ the former condition tends to the latter. We note that it is possible for the addition of an adiabatically mixed surface

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layer to make lee waves possible when they would not occur otherwise, provided the adiabatic layer is not added at the expense of the intermediate layer. The effect on the maximum amplitude of the additional layer can be seen in Fig. 5 which contains amplitude and wavelength curves for $L = 2\ km^{-1}$ and $l_s = 0.2\ km^{-1}$. This figure contains dashed curves computed for an airstream with a low level adiabatic layer of depth 0.5 km and full curves applicable when the adiabatic layer is absent. The general conclusions reached for the two layer model are seen to be equally applicable if there is an additional layer at the surface having an adiabatic lapse rate, the only differences being a limited reduction in amplitude and an increase of wavelength throughout the range of values.

6. Conclusion

Over the British Isles airstreams which formally satisfy the theoretical conditions for lee waves are probably very common, especially in winter. However, the amplitude of such waves will be subject to very wide variations even amongst airstreams having very similar profiles of l^2 because of the critical way in which the amplitude depends on the airstream characteristics. The largest amplitude waves occur when the airstream satisfies the condition for waves by only a small margin, and in this region very large changes of amplitude may result from very small changes in the airstream characteristics. Apart from the question of this sensitive region, it may be said that larger amplitude waves are theoretically more likely in airstreams containing a shallow layer of great stability than in conditions of slight stability through a deep layer. The main effect of an adiabatically mixed layer near the ground is to reduce the amplitude and increase the wavelength.

Quite apart from the variations which depend on the airstream characteristics it appears that a given airstream is sensitive to the scale of the disturbing mountain. If the mountain is made broader or narrower than the optimum width for the prevailing airstream a considerable reduction of wave amplitude is to be expected.

References

Förchtgott, J.	1949	Bull. Met. tchécosl. Prague, 3, p.49.
Pilsbury, R.K.	1955	Met. Mag. (in press).
Scorer, R.S.	1949	Quart. J.R. Met. Soc., 75, p.41.
	1953	Ibid., 79, p.70.

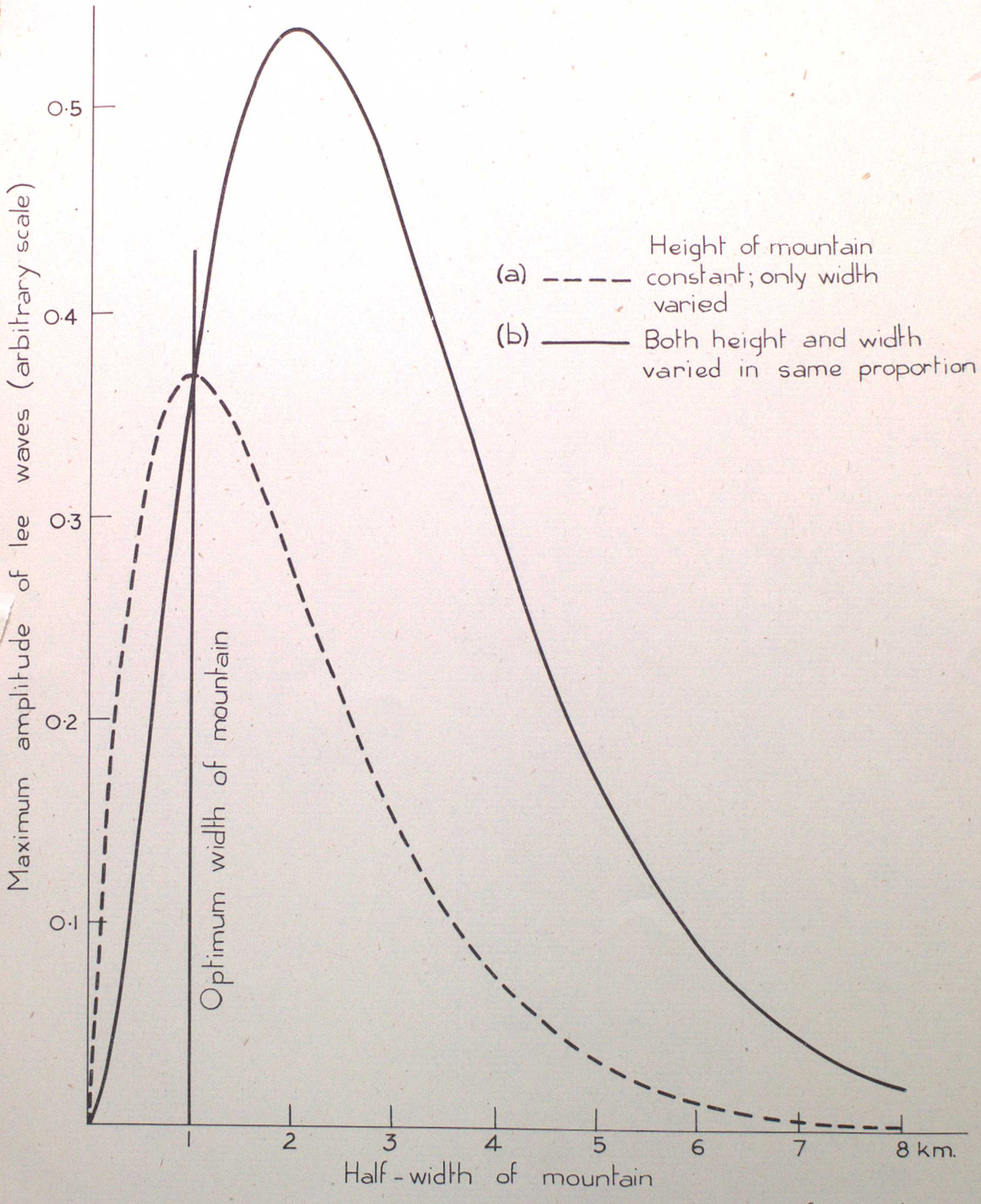

Fig.1 Variation of lee wave amplitude with size of mountain
$k = 1\ km^{-1}$

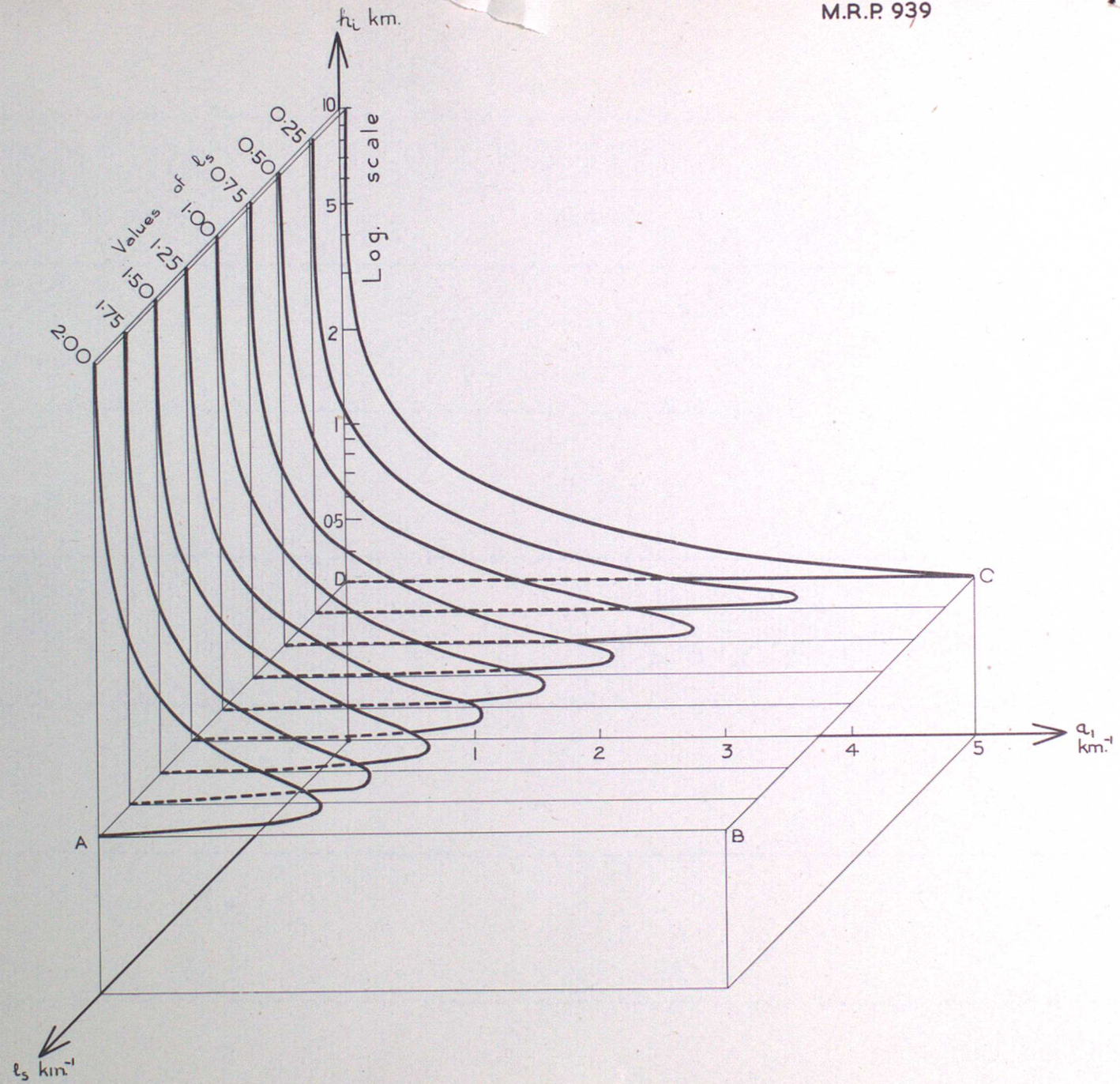

Fig.2 Variation of lee wave amplitude (a_1) with airstream characteristics
L = constant = 5 km.$^{-1}$

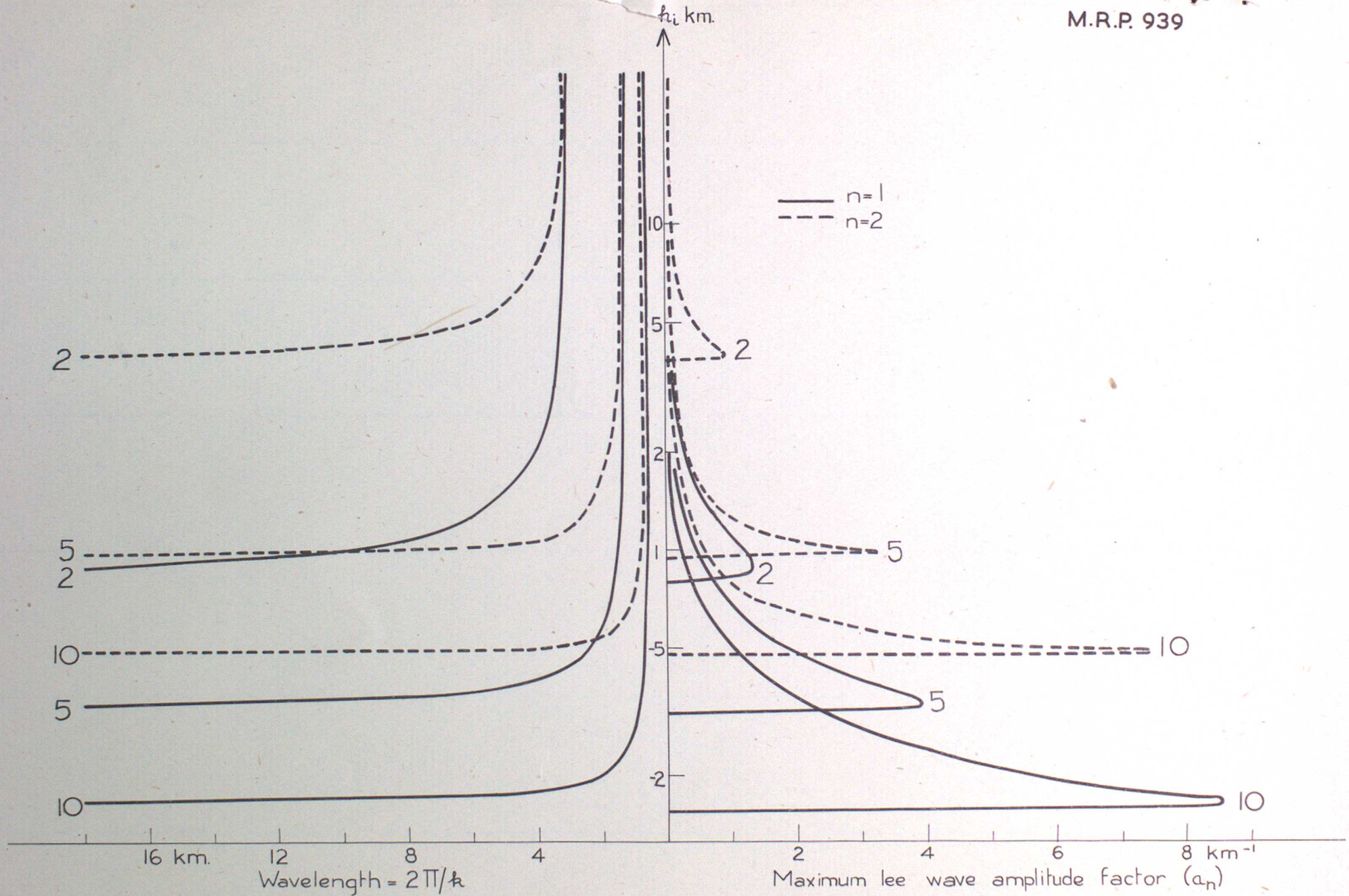

Fig.3 Variation of lee wave amplitude and wavelength with airstream characteristics
l_s = constant = 0·2 km^{-1} Values of L (2, 5, 10 km^{-1}) marked against curves

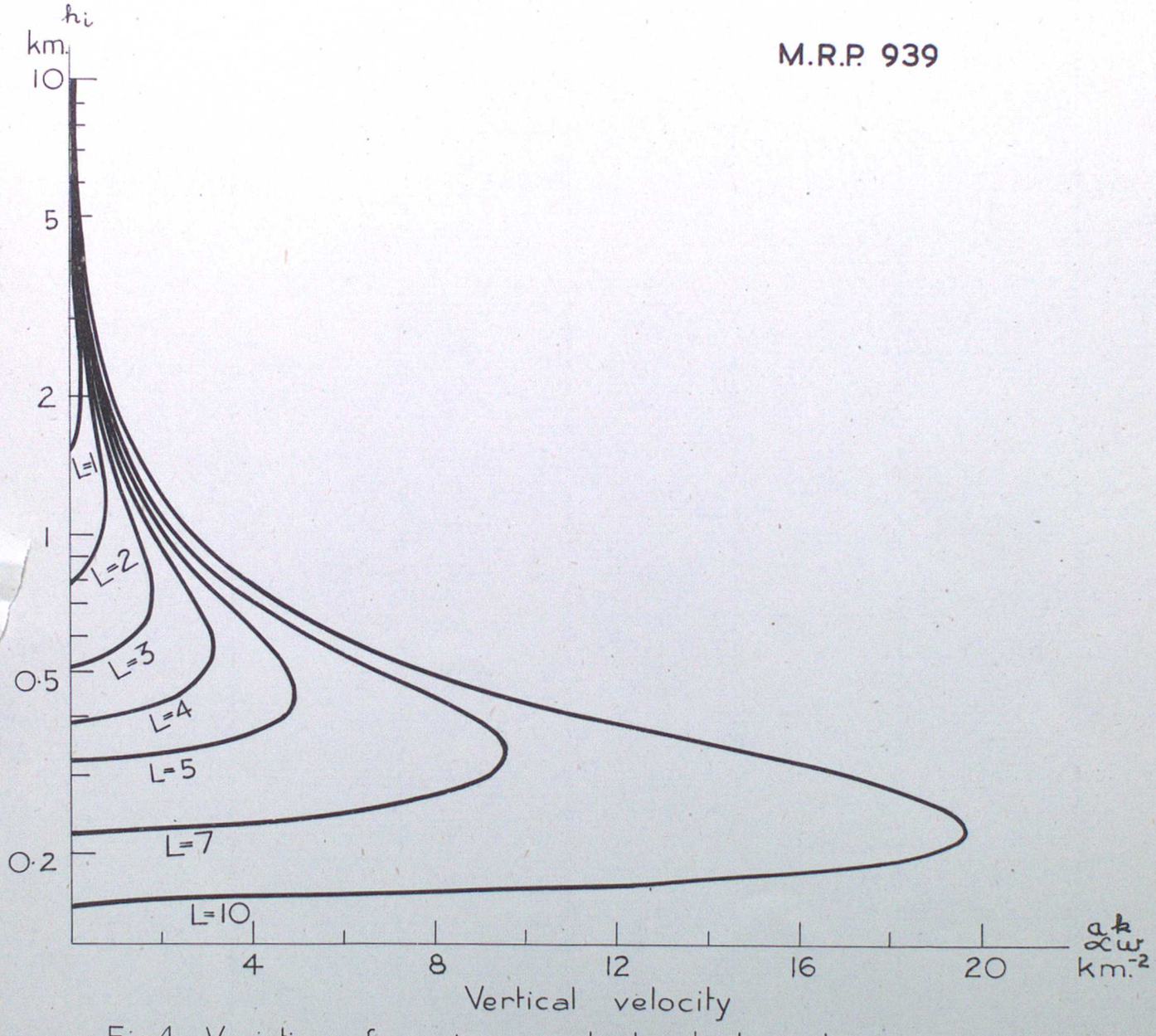

Fig 4 Variation of maximum vertical velocity in lee waves with airstream characteristics. Values of L are indicated. Results are independent of l_s

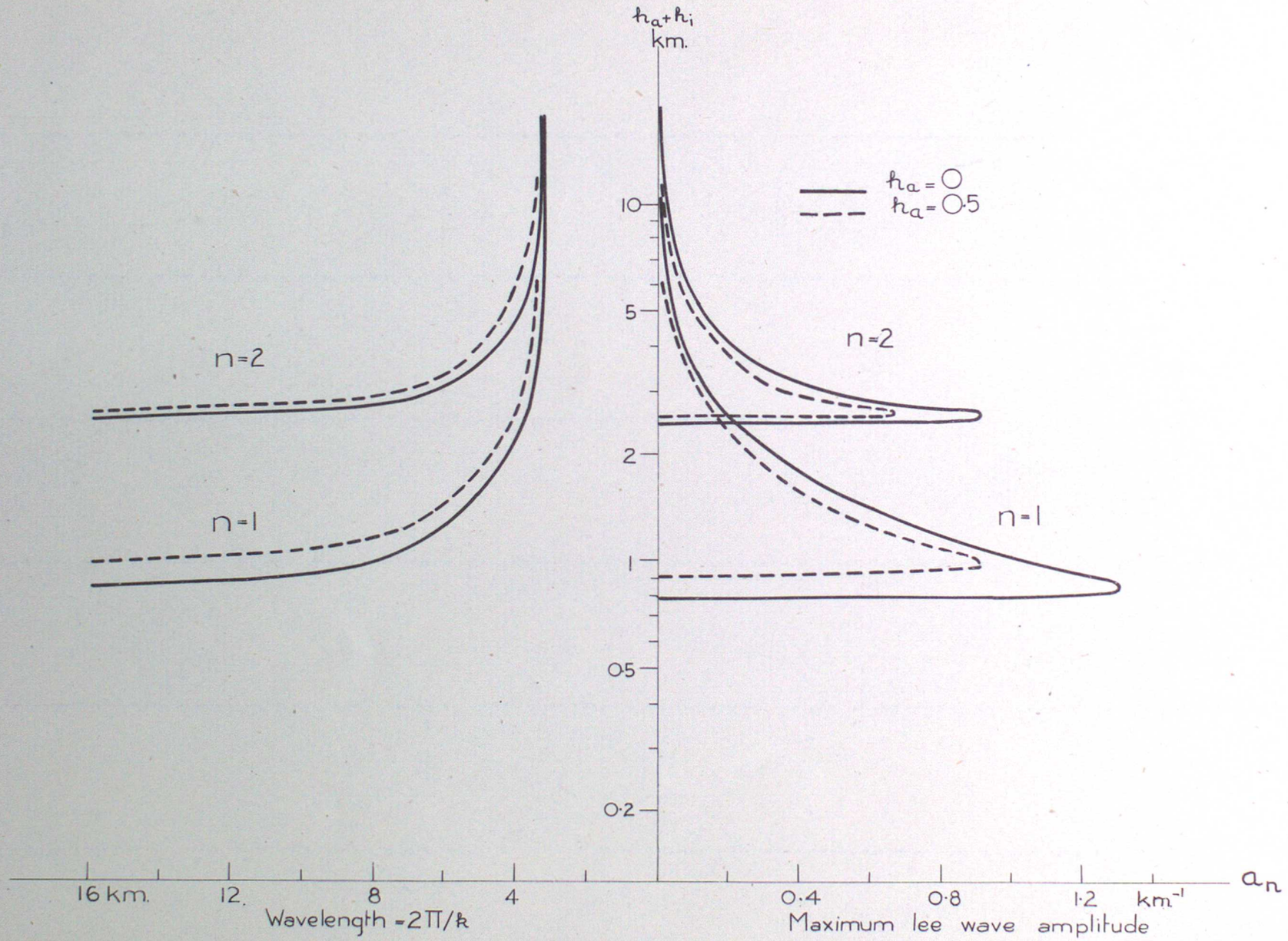

Fig 5. Variation of lee wave amplitude and wavelength for 3-layer airstream
$L = 2\ km.^{-1}$ $l_s = 0.2\ km.^{-1}$